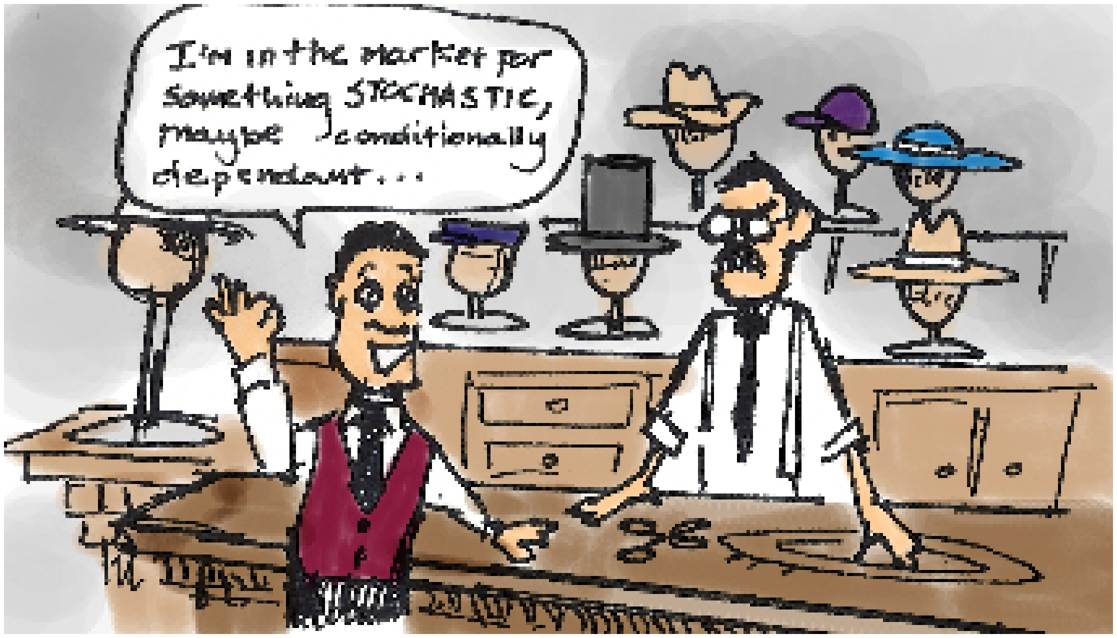
Aaron Brown

Opening Pandora's Box

Taking a look at Kent Osband's long-awaited follow-up to *Iceberg Risk* inspires an excursion into problems of correlation



Wilmott readers are of course familiar with Kent Osband's frequent contributions. By the time you are reading this, his new book, *Pandora's Risk*, should be available. I managed to score a prepublication review copy. The book is a delight on three levels. Like his earlier *Iceberg Risk*, it is built around an amusing story with dialog that explores some deep ideas about risk. In addition, there is brilliant and unconventional exploration of important ideas in finance, including money, risk, uncertainty, credit, credit ratings, securitization, and VaR.

Finally, there is enough clever modeling to make this almost a recreational mathematics book. Kent illustrates his ideas with simple mathematical models that have surprising properties. An extensive set of appendices add refinements to each one. Even if you decide the models give only limited insight to the economic phenomena they are meant to illuminate, they are a pleasure to play with for their own sakes.

For example, Chapter 9, *Insecuritization*, discusses how and why people are misled by looking at standard deviation and correlation to underestimate tail risk. The chapter takes up where *Iceberg Risk* left off, showing why fat-tailed approximations and copulas miss the point. It treats specifically various models used by Moody's to evaluate structured products. There's a lot of important, actionable finance here, and some useful mathematics as well.

However, being a dilettante, I was distracted by the discussion of Osband's triangle, exchangeability, and mixing distributions. This is just fun math. I came up with the following example on my own; Kent bears no responsibility for it.

Suppose you have a portfolio of eleven bonds, each with a 10 percent probability of default, and the defaults are uncorrelated. What is the probability of zero defaults? One default? All eleven defaulting?

If you're careless, you'll answer $0.9^{11} = 31\%$ for zero, $11 \times 0.9^{10} \times 0.1 = 38\%$ for one, and $0.1^{11} = 0.000000001\%$ for ten. These computations are correct if the defaults are independent. But that's not what uncorrelated means. Two events are uncorrelated if the probability of both occurring is equal to the product of each event occurring. In this case, it means we know the chance of any specific pair of bonds defaulting is $0.1^2 = 1\%$.

A good way to think about this is to imagine a hat with 100 slips of paper inside. One piece will be picked and whatever bonds are listed on it will default. The 10 percent probability for each bond defaulting means each bond is listed on exactly ten of the slips. The 1 percent probability of any pair means that each pair of bonds is listed on exactly one piece of paper. There are 560 ways to satisfy these conditions, treating the bonds as indistinguishable, including:

- Each possible pair of bonds is listed on individual slips, 55 in all, and the other 45 slips are blank.
- All eleven bonds are listed on one slip, each bond is listed alone on nine slips, and no slips are blank.

In the first case the chance of zero defaults is 45 percent, two defaults is 55 percent, and any other number, including one and eleven defaults, is 0 percent. In the second case the chance of one default is 99 percent, ten defaults is 1 percent, and any other number is 0 percent. Many other situations are possible.

Suppose you are thinking of writing a CDS on this portfolio that will pay $1,000 for each default above three over the next year. If the bond defaults were independent, the expected payment under this contract would be $22. If instead you assume the defaults are modeled with the slips of paper in the hat, and that all 560 possible arrangements with 10 percent default probability for each bond and zero correlation between any two bonds are equally likely, then the expected payment is $58.

It turns out that the independence assumption is very special; there are relatively few assignments of bonds to slips that produce an expected payment less than $22. These are ones with lots of three-default slips, such as 27 blank slips, 54 slips with single names, one slip with two names, 18 slips with three names, and no slips with more than three names. Most of the assignments have much more risk of extreme events, up to an expected $110 payout on the CDS (one blank slip, 84 slips with single names, four slips with five names each, one slip with six names). And the few examples that give low expected payouts for specific CDS will give extremely high expected payouts for other CDS. Independence is not some neutral assumption, equally likely to over- or underestimate the chance of extreme events; it's a very strong assumption that in practice always means dangerous neglect of important risks.

Observation is little help, in fact it's usually misleading. Suppose you draw slips, with replacement, from the hat for five years. We assume you have a new set of eleven bonds each year and a new hat with identical slips. The most likely outcome is that you see five defaults over five years. That happens 24 percent of the time. Conditional on that observation, you expect to pay $64 on the CDS next year. Any other number of observed defaults from zero to eleven leads you to expect a lower payout. In other words, the most likely observation gives you the greatest probability of future extreme observations; while extreme observations are actually evidence of safety. Only if you observe more than eleven defaults, which happens only 1.8 percent of the time, would you expect to pay more than $64 on the CDS.

Correlation tells you only about pairs of defaults, it says nothing about the probability or improbability of larger numbers. However, people often use correlation to make predictions about more than two defaults. In 1931, the brilliant Italian statistician Bruno de Finetti proved a remarkable result to make this easier. If events are "exchangeable" (meaning for any k, the distribution of the number of defaults is the same for any subset of k bonds), the number of defaults can be described as a mixture of conditionally independent binomial distributions. I won't go into the technical details of what that means, but in practice we can treat the problem as two draws. First we draw a probability, then each bond either defaults or not independently. Bond defaults are dependent, because the selection of the probability affects them all, but they are conditionally independent, because once the probability is drawn, bond defaults are no longer dependent. Unfortunately, conditional independence is just as special as independence, and just as dangerous. This is the problem with assuming multivariate normality or using methods optimized for multivariate normality.

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Suppose instead of zero correlation in the example above, I told you the correlation between any two bonds was 0.25. One possible generating mixture is to say there is a 10 percent chance that all bonds will default with independent 55 percent probability and a 90 percent chance that all bonds will default with independent 5 percent probability. That preserves the 10 percent unconditional probability of default ($0.1 \times 0.55 + 0.9 \times 0.05 = 0.1$). The probability of any pair of bonds defaulting is 3.25 percent ($0.1 \times 0.55^2 + 0.9 \times 0.05^2 = 0.0325$, which implies a 0.25 correlation).

If this is the case, Table 1 shows the probabilities of different numbers of defaults. This looks more reasonable than the extreme cases above. But it's still a little funny; there is more chance of six defaults than four, for example.

Basel II specifies an Advanced Internal Ratings based methodology that assumes the mixing distribution (the distribution that selects

Table 1: Probability of various numbers of defaults under a 90%/10% mixing distribution

Number of defaults	Probability
0	51%
1	30%
2	7.9%
3	1.7%
4	1.3%
5	1.9%
6	2.4%
7	2.1%
8	1.3%
9	0.51%
10	0.13%
11	0.01%

Table 2: Probability of various numbers of defaults under three mixing distributions with the same probability of default and correlation

Number of defaults	Basel mixture	Mixture 1	Mixture 2
0	19%	36%	53%
1	58%	38%	16%
2	19%	18%	12%
3	4.0%	5.3%	10%
4	0.7%	1.0%	5.8%
5	0.10%	0.14%	2.4%
6	0.01%	0.01%	0.71%
7	0.00%	0.00%	0.15%
8	0.00%	0.00%	0.02%
9	0.00%	0.00%	0.00%
10	0.00%	0.00%	0.00%
11	0.00%	1.3%	0.00%

the probability of default) is based on a Gaussian distribution. That gives even smoother predictions. For credits with 10 percent probability of default, Basel specifies a 0.121 correlation. Table 2 shows the Basel prediction (based on a one-factor Vasicek model) along with two other predictions using the same 10 percent probability of default and 0.121 correlation, but different mixing distributions. Mixture 1 is a 1.3 percent probability of 100 percent probability of default and a 98.7 percent probability of 8.8 percent probability of default. Mixture 2 is a 40 percent probability of 22.8 percent probability of default and a 60 percent probability of 1.5 percent probability of default.

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As you can see, the predictions are radically different. We are led to suspect that correlation doesn't tell us much about the distribution of defaults. The problem is worse than I have described. There are many other mixing distributions that could give even more extreme results. And defaults are not exchangeable, so we don't even know we have a mixing distribution.

Worse still, we don't know the correlation, or even the probability of default with any precision. The correlations are likely different for different pairs of credits. We have to observe some of the more likely outcomes, like zero, one or two defaults, and try to guess the probability of less likely outcomes in the future. And however much data we have, there's always the possibility that parameters have changed or that the current set of bonds is not similar to the ones used to form our estimates.

As a side point, it's not even true that people all define credit correlation in the same way. In the structured credit world, the most common definition of correlation is correlation of time to default, and other definitions are used as well. There's no way to convert from one type of correlation to another, they are entirely different concepts.

Okay, so we're back with the hat. We watch the draws for a while and see lots of blank slips and a reasonable number of slips with one or two defaults, rarely more. That makes it unlikely that there are lots of slips of paper with lots of defaults on them. But there could easily be a non-negligible number of slips that imply mass defaults. Estimating a correlation coefficient from the slips we have drawn is clearly a pointless exercise that tells us nothing about the slips remaining in the hat.That doesn't mean correlation cannot help us, it can. But it won't tell us the likelihood of extreme events. We'll have to estimate that from fundamental analysis and set up contingency plans if events deviate from prediction. In technical terms, we make a guess and find a good hiding place.

The appropriate concept of correlation is helpful in evaluating our risk and hedging it. Let's begin by considering one extreme. Every credit in the world can be assigned a unique serial number from 1 to the number of credits, and they default in that order. We don't know the probability of default, but we do know that if credit number N goes, all credits with serial numbers lower than N also go. In that case, if we know how many defaults are on the slip drawn from the hat, we know exactly which credits in our portfolio will default, all the ones with serial number less than the number of defaults on the slip. If we can buy derivatives that pay off based on the number of defaults on the slip, we can hedge our portfolio perfectly. The cost of hedging is a measure of the risk of the portfolio.

Now let's go to the other extreme, credits are exchangeable in de Finetti's terms. Define p as the number of defaults on the slip to the total number of credits. Each credit in our portfolio will default independently with probability p. Now we can't have a perfect hedge, but we can limit a percentile of our losses. For example, we could make sure there was less than a 0.1 percent chance of more than three defaults. Again, the cost of doing this would be a measure of the risk of the portfolio.

The first extreme described above is correlation one. It's not that either every credit pays or no credit pays, it's that the number of defaults completely determines the identity of the defaults. This is the correlation we should care about. The second extreme is correlation zero. Given N defaults, any subset of N credits is equally likely to be the defaulting set.

The truth, of course, is in between. We know CCC credits are more likely to default than AA ones, but it isn't true that every BB credit has to default before any BBB credit does. Correlation in this sense is fairly easy to estimate, at least if we have a history of consistent ratings with a reasonably large number of credits that covers at least three market cycles. The correlation need not depend only on rating, it can depend on type of security, industry, or anything else for which we can get sufficient data. All that matters is that we can sort all credits into buckets such that if we know the number of defaults within the bucket, we have no strong reason to guess one subset of credits over another as the guilty parties. If we know the relative average default probabilities among the buckets we can refine our predictions.

Now we have a double conditioning model. Given the number of defaults on the slip of paper, we can compute the distribution of default frequency within each bucket. Given the frequency within each bucket, we can compute the distribution of our portfolio losses. If we can trade derivatives based on the total number of defaults, we can hedge our portfolio, with some error.

Most of the traditional uses of the concept of correlation in credit are misleading, and more likely to encourage error than generate insight. Only by thinking more deeply about what is a useful definition of correlation will we get more good than harm out of it. The model I sketched above is by no means the only way to use the concept of correlation well. So don't be afraid to use correlation in your credit modeling, but make sure it is good correlation. W